ETH zürich
Institute of Theoretical Computer Science
Prof. Karl Bringmann, Prof. Rasmus Kyng, Prof. Vera Traub
Manuel Christalla, Christoph Grunau, Paul Paschmanns

Algorithms and Probability **Session Sheet S5** **FS 2026**

For the exercise sessions on 19 March 2026.

Exercise S5.1 – *3-colorable graphs*

In the lecture you have seen an algorithm that finds an $O\left(\sqrt{|V|}\right)$-coloring for every 3-colorable graph (V, E). Fix $\alpha = \sqrt{|V|}$. The algorithm goes as follows.

1. While there is a vertex v with degree at least α, color v with a new color, color the neighbors of v with two additional new colors, and delete v and all its neighbors from the graph.
2. Color all remaining vertices with at most $\alpha + 1$ colors.

We want to analyze and generalize this algorithm.

(a) Where in the algorithm do we use that the graph is 3-colorable?

(b) How many colors do we need at most (depending on α)? Show that for $\alpha = \sqrt{|V|}$ we only need $O\left(\sqrt{|V|}\right)$ colors.

(c) Show that your bound in (a) is tight. More precisely, create a 3-colorable graph on which the algorithm uses $\Omega\left(\sqrt{|V|}\right)$ colors.

(d) Can you describe an algorithm that finds a $O\left(|V|^{\frac{2}{3}}\right)$-coloring for every 4 colorable graph?

(e) Can you describe an algorithm that finds a $O\left(|V|^{\frac{q-2}{q-1}}\right)$-coloring for every q colorable graph, where $q \geq 2$ is a constant?

Solution S5.1 – *3-colorable graphs*

(a) We use that the graph is 3-colorable to ensure that the neighbors of each vertex can be 2-colored.

(b) Let k be the number of iterations that the while loop is executed. For each iteration, we need 3 colors. Thus, we need $3k$ colors for all iterations of the while loop. In each iteration of the while loop, we delete at least $\alpha + 1$ vertices. Hence, it is executed at most $\frac{n}{\alpha+1} \leq \frac{n}{\alpha}$ times. Hence, we need at most $3\frac{n}{\alpha}$ colors during the while loop. Afterwards, each remaining vertex has degree strictly less than α. Thus, the remaining vertices can be colored with at most $\alpha + 1$ colors, e.g. using the greedy algorithm (if α is an integer, α many colors suffice). In total this gives us an upper bound of $3\frac{n}{\alpha} + \alpha + 1$ on the colors we need. For $\alpha = \sqrt{|V|}$ this equals $4\sqrt{|V|} + 1$.

(c) Fix sone constant β. A wheel of size 2β is a cycle of length 2β and an additional "center vertex" that has one edge to each vertex on the cycle. Note that every wheel is 3-colorable. Let G be a graph consisting of β many disjoint wheels of size 2β. The graph G has $\beta(2\beta+1)$ many vertices. Each center vertex has a degree of $2\beta > \sqrt{\beta(2\beta+1)}$. Hence, when applied to G, the algorithm will execute β many iterations of the while loop (one for each "center vertex"). In each iteration, it introduces 3 new colors. Hence, it uses at $3\beta = \Omega\left(\sqrt{|V|}\right)$ colors.

(d) Set $\alpha = |V|^{2/3}$.

(a) While there is a vertex v with degree at least α, color v with a new color, color α many of the neighbors of v with $4\sqrt{\alpha}+1$ additional new colors (using our algorithm for 3-colorable graphs), and delete v and all its neighbors from the graph.

(b) Color all remaining vertices with at most $\alpha + 1$ colors.

Similar to before, let k be the number of iterations of the while loop. The number of colors used can be bounded by

$$k \cdot \left(4\sqrt{\alpha}+1\right) + \alpha + 1 \leq \frac{n}{\alpha} \cdot \left(4\sqrt{\alpha}+1\right) + \alpha + 1\,.$$

For $\alpha = |V|^{2/3}$, this value is at most $O\left(|V|^{\frac{2}{3}}\right)$.

(e) One can proceed similarly to (d). To get a q-coloring, one calls the algorithm to obtain $q-1$-coloring. The value α can be chosen as $\alpha = O\left(|V|^{\frac{q-2}{q-1}}\right)$.