Beweis durch vollständige Induktion

Idee: Beweis einer mathematischen Aussage für den kleinstmöglichen Fall und basierend darauf Gültigkeit für den allgemeinen Fall zeigen.

Beispiel: $1+2+\ldots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

Base Case: $n=1$ $\quad \sum_{i=1}^{1} i = \frac{1(1+1)}{2} = 1$

Induktionshypothese: $\sum_{i=1}^{k} i = \frac{k(k+1)}{2}$ gilt für manche $k \geq 1$.

Induktionsschritt: $\sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1) \overset{\text{I.H.}}{=} \frac{k(k+1)}{2} + (k+1) = \frac{k(k+1)}{2} + \frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$

(This subtask is from August 2019 exam). Let $T : \mathbb{N} \to \mathbb{R}$ be a function that satisfies the following two conditions:

$$T(n) \geq 4 \cdot T(\tfrac{n}{2}) + 3n \qquad \text{whenever } n \text{ is divisible by 2;}$$
$$T(1) = 4.$$

Prove by mathematical induction that

$$T(n) \geq 6n^2 - 2n$$

holds whenever n is a power of 2, i.e., $n = 2^k$ with $k \in \mathbb{N}_0$.

Tipp: Über k iterieren!

Base Case $k=0$: $n=1$, $T(1) = 4 \geq 6 \cdot 1^2 - 2 \cdot 1 = 4$

Induktionshypothese: $T(m) \geq 6m^2 - 2m$ gilt für manche $m = 2^\ell$ mit $\ell \in \mathbb{N}_0$.

Induktionsschritt: $T(2^{\ell+1}) = T(2m) \geq 4 \cdot T(m) + 6n$

$$\begin{aligned}
&\overset{\text{I.H.}}{\geq} 4 \cdot (6m^2 - 2m) + 6m \\
&= 6 \cdot 4m^2 - 2m \\
&= 6 \cdot (2m)^2 - 2m \\
&\geq 6 \cdot (2m)^2 - 2 \cdot (2m)
\end{aligned}$$

Durch vollständige Induktion konnte diese Aussage für alle $n = 2^k, k \in \mathbb{N}$, gezeigt werden.

Asymptotisches Wachstum

Definition 1. Let $f, g : \mathbb{N} \to \mathbb{R}^+$ be two functions. We say that f *grows asymptotically faster than* g if $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$.

This definition is also valid for functions defined on $\mathbb{R}^+$ instead of $\mathbb{N}$. In general, $\lim_{n\to\infty} \frac{g(n)}{f(n)}$ is the same as $\lim_{x\to\infty} \frac{g(x)}{f(x)}$ if the second limit exists.

(a) $f(n) := n \log n$ grows asymptotically faster than $g(n) := n$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n}{n\log n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

(b) $f(n) := n^3$ grows asymptotically faster than $g(n) := 10n^2 + 100n + 1000$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{10n^2+100n+1000}{n^3} = \lim_{n\to\infty} \frac{\cancel{n^3}\left(\frac{10}{n} + \frac{100}{n^2} + \frac{1000}{n^3}\right)}{\cancel{n^3}} = \lim_{n\to\infty} 0+0+0 = 0$$

(c) $f(n) := 3^n$ grows asymptotically faster than $g(n) := 2^n$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{2^n}{3^n} = \lim_{n\to\infty} \left(\tfrac{2}{3}\right)^n = 0$$

Theorem 1 (L'Hôpital's rule). *Assume that functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ are differentiable, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ and for all $x \in \mathbb{R}^+$, $g'(x) \neq 0$. If $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = C \in \mathbb{R}_0^+$ or $\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \infty$, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

(a) $f(n) := n^{1.01}$ grows asymptotically faster than $g(n) := n \ln n$.

$$\lim_{n\to\infty} \frac{n \ln n}{n^{1{,}01}} = \lim_{n\to\infty} \frac{\ln n}{n^{0{,}01}} \overset{H.}{=} \lim_{n\to\infty} \frac{\frac{1}{n}}{0{,}01 \cdot n^{-0{,}99}} = \lim_{n\to\infty} \frac{1}{0{,}01\, n^{0{,}01}} = 0$$

(c) $f(n) := e^n$ grows asymptotically faster than $g(n) := n^2$.

$$\lim_{n\to\infty} \frac{n^2}{e^n} \overset{H.}{=} \lim_{n\to\infty} \frac{2n}{e^n} \overset{H.}{=} \lim_{n\to\infty} \frac{2}{e^n} = 0$$

Trick:

$$x^a = \left(e^{\ln(x)}\right)^a = e^{\ln(x)\cdot a}$$

(d)* $f(n) := 1.01^n$ grows asymptotically faster than $g(n) := n^{100}$.

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{n^{100}}{1{,}01^n} = \lim_{n\to\infty} \frac{e^{\ln(n)\cdot 100}}{e^{\ln(1{,}01)\cdot n}} = \lim_{n\to\infty} e^{\ln(n)\cdot 100 - n\cdot \ln(1{,}01)} = \lim_{n\to\infty} e^{\text{"}-\infty\text{"}} = 0$$

$$\lim_{n\to\infty} \ln(n)\cdot 100 - n\cdot \ln(1{,}01) = \underbrace{\lim_{n\to\infty} n}_{\to\infty} \cdot \lim_{n\to\infty} \underbrace{\left(\underbrace{\frac{100\ln(n)}{n}}_{\to 0} - \underbrace{\ln(1{,}01)}_{>0,\ <0}\right)}_{<0} = -\infty$$

(e) $f(n) := \log_2 n$ grows asymptotically faster than $g(n) := \log_2 \log_2 n$.

$\log_2 x = \frac{\ln(x)}{\ln(2)}$

$$\lim_{n\to\infty} \frac{g(n)}{f(n)} = \lim_{n\to\infty} \frac{\log_2 \log_2 n}{\log_2 n} \overset{*}{=} \lim_{y\to\infty} \frac{\log_2 y}{y} = \lim_{y\to\infty} \frac{\ln(y)}{\ln(2)\cdot y} \overset{H.}{=} \lim_{y\to\infty} \frac{\frac{1}{y}}{\ln(2)} = 0$$

* Substitution mit $y = \log_2 n$

Exercise 0.4 *Simplifying expressions.*

Simplify the following expressions as much as possible without changing their asymptotic growth rates.

Concretely, for each expression $f(n)$ in the following list, find an expression $g(n)$ that is as simple as possible and that satisfies $\lim_{n\to\infty} \frac{f(n)}{g(n)} \in \mathbb{R}^+$.

(a) $f(n) := 5n^3 + 40n^2 + 100$

$g(n) = n^3$

(c) $f(n) := n \ln n - 2n + 3n^2$

$g(n) = n^2$

(d) $f(n) := 23n + 4n \log_5 n^6 + 78\sqrt{n} - 9$

$\ln(x^a) = a\ln(x)$

$g(n) = n \ln n$

$4n \log_5 n^6 = 24n \log_5 n = 24n \frac{\ln(n)}{\ln(5)} = \frac{24}{\ln(5)} n \ln(n)$

(e) $f(n) := \log_2 \sqrt{n^5} + \sqrt{\log_2 n^5}$

$\log_2 \sqrt{n^5} = \log_2 n^{\frac{5}{2}} = \frac{5}{2} \log_2 n = \frac{5}{2} \cdot \frac{1}{\ln(2)} \cdot \ln(n)$

$\sqrt{\log_2 n^5} = \sqrt{5 \cdot \frac{1}{\ln(2)} \cdot \ln(n)}$

$g(n) = \ln(n)$

(f)* $f(n) := 2n^3 + (\sqrt[4]{n})^{\log_5 \log_6 n} + (\sqrt[7]{n})^{\log_8 \log_9 n}$

$\log_a(x) \leq \log_b(y)$ wenn $x \leq y$ und $a \geq b$

$\lim_{n\to\infty} \frac{(\sqrt[7]{n})^{\log_8 \log_9 n}}{(\sqrt[4]{n})^{\log_5 \log_6 n}} = \lim_{n\to\infty} \frac{n^{\frac{1}{7}\log_8 \log_9 n}}{n^{\frac{1}{4}\log_5 \log_6 n}} = \lim_{n\to\infty} n^{\frac{1}{7}\log_8 \log_9 n - \frac{1}{4}\log_5 \log_6 n}$

$\frac{1}{7} \log_8 \log_9 n - \frac{1}{4} \log_5 \log_6 n$

$\underbrace{\log_9 n}_{x} \leq \underbrace{\log_6 n}_{y}$

$\log_8 \log_9 n \leq \log_5 \log_6 n$

$g(n) = n^{\frac{1}{4}\log_5 \log_6 n}$

Zusätzlicher Induktionsbeweis aus der Prüfung aus FS21:

Theory Task T2. / 18 P

In this part, you should **justify your answers briefly**, e.g., by sketching the derivation.

/ 3 P a) *Induction:* Prove by induction that $n^3 \geq 4n^2 + 25$ holds for $n \geq 5$.

Lösung:

Base Case: $n=5$ $\quad 4 \cdot 5^2 + 25 = 5^3 = 125$ ✓

Induktionshypothese: $m^3 \geq 4m^2 + 25$ hält für manche $m \geq 5$

Induktionsschritt:
$$\begin{aligned}(m+1)^3 &= m^3 + 3m^2 + 3m + 1 \\ &\overset{\text{I.H.}}{\geq} 4m^2 + 25 + 3m^2 + 3m + 1 \\ &= 7m^2 + 3m + 26 \\ &\geq 7m^2 + 2m + 29 \\ &\geq 4m^2 + 8m + 29 \\ &= 4(m^2 + 2m + 1) + 25 \\ &= 4(m+1)^2 + 25\end{aligned}$$

hier nutzt man $m \geq 5$

$n^2 \geq 4n^2 + 25$ hält für alle $n \geq 5$. ∎