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Algorithmen und Wahrscheinlichkeit
Exercise Session 3

Exercise 1 – *Quantitiative Hall's Theorem*

For a bipartite graph $G = (A \cup B, E)$, with $|A| = |B| = n$, we define a *defficiency* of G ($\mathrm{deff}(G)$) to be a maximum over $X \subset A$ of $|X| - |N(X)|$. The Hall's theorem states that G has a perfect matching if and only if the defficiency of G is 0. We will show a quantitative version of Hall's theorem, using the Hall's theorem itself.

Specifically, show that the size of the largest (in terms of the number of edges) matching M in the bipartite graph G is equal to $n - \mathrm{deff}(G)$.

1. Show that if $\mathrm{deff}(G) = k$ then every matching M has size $|M| \leq n - k$. (*Hint:* this can be done with a direct argument.)
2. Show that when $\mathrm{deff}(G) \leq k$, there is a matching M in G of size at least $n - k$. (*Hint:* modify the graph G by adding k vertices on each side, and apply Hall's theorem to the modified graph G'.) (*If you feel that this is a bit too difficult to solve for your group during the exercise session, you can split this into easier sub-problems — explain the construction of the graph G' from the solution to students, and ask them to show the Hall's condition, and deduce what we want from Hall's theorem*)

Solution 1

1. Consider a matching M, and a set $X \subset A$ with $|X| - |N(X)| \geq k$. Note that matching M has at most $|N(X)|$ edges incident with vertices in X. The number of edges incident with vertices in $A \setminus X$ is clearly at most $|A \setminus X| = n - |X|$. As such, the total number of edges in M is $|M| \leq |N(X)| + n - |X| \leq n - k$.
2. Take a graph $G' = (A' \cup B', E')$ obtained by adding k vertices on each side in G. That is $A' = A \cup A_1$ and $B' = B \cup B_1$, where $|A_1| = |B_1| = k$, and we connect all vertices from A_1 with all vertices from B'; as well as we connect all vertices from B_1 with all vertices from A'. We will argue that G' satisfies the Halls condition. Indeed, for any set $X \subset A'$, either $X \cap A_1 \neq \emptyset$, in which case $|N_G(X)| = |B'| = n + k \geq |X|$. or $X \subset A$, in which case $|N_{G'}(X)| = |N_G(X) \cup B_1| = |N_G(X)| + k \geq |X| - \mathrm{deff}(G) + k = |X|$.

   Applying Hall's theorem now, we conclude that G' has a perfect matching M'. In the perfect matching M' at most k vertices from A is matched to vertices in B_1 (since $|B_1| = k$), so the remaining $n - k$ vertices are matched through M' to some vertices in B.

Exercise 2 – *Köning's Theorem*

For a graph $G = (V, E)$, we say that a subset of vertices $S \subset V$ is a *vertex cover* of G if every edge $e \in E$ is incident to at least one vertex in S. Equivalently, S is a vertex cover of G, if the induced graph $G[V \setminus S]$ has no edges.

Using the quantitative version of Hall's Theorem (Exercise 1), we will show that in a bipartite graph $G = (A \cup B, E)$ the size of the smallest vertex cover is equal to the size of the largest matching in G.

1. Show that if S is any vertex cover in G, and M is a matching in G, then $|M| \leq |S|$.
2. Show that if $\mathrm{deff}(G) = k$, there is a vertex cover of size $n - k$. Conclude that the size of the largest matching is equal to the size of the smallest vertex cover.

Solution 2

1. By the definition of vertex cover, each edge of M is incident with at least one vertex from S. Consider a mapping $f : M \to S$ mapping each edge $m \in M$ to a vertex $s \in S$ incident with m. By the definition of matching, every vertex of S is incident with at most one edge of M, therefore f is a one-to-one function, implying that $|M| \leq |S|$.
2. Let $X \subset A$ be a set with $|X| - |N(X)| = k$. Then $S := (A \setminus X) \cup N(X)$ is a vertex cover of S (for any edge incident with X it is covered by $N(X)$, and remaining edges are covered by $A \setminus X$). Moreover $|S| = n - |X| + |N(X)| = n - k$.

Exercise 3 – *Traveling Salesman Problem*

We consider a Traveling Salesman Problem on a complete graph K_n, where the cost function $\ell : \binom{[n]}{2} \to \mathbb{R}$ satisfies a weaker version of triangle inequality. Specifically, assume that for every $x, y, z \in [n]$ we have

$$\ell(x, z) \leq 1.1(\ell(x, y) + \ell(y, z)).$$

Show that the algorithm presented in the lecture provides $O(n^{\log_2(1.1)})$-approximation for the TSP problem.

Solution 3

We can show by induction that for any path $v_0, v_1, \ldots v_2, \ldots v_j$ of length j, s.t. $2^k < j \leq 2^{k+1}$, we have

$$\ell(v_0, v_j) \leq 1.1^{k+1} \sum_{i=0}^{j-1} \ell(v_i, v_{i+1}),$$

by splitting the path $v_0, \ldots v_j$ into two parts $v_0, \ldots v_{\lfloor j/2 \rfloor}$ and $v_{\lfloor j/2 \rfloor}, \ldots, v_j$. Since any path without repeated vertices in a graph G has at most $n - 1$ edges, we have for any path without repeated vertices

$$\ell(v_0, v_j) \leq 1.1^{\lceil \log_2 n \rceil} \sum_{i=0}^{j-1} \ell(v_i, v_{i+1}) \leq O(n^{\log_2 1.1}) \sum_{i=0}^{j-1} \ell(v_i, v_{i+1}).$$

Together with the rest of the argument presented during the lecture, this implies a $O(n^{\log_2 1.1})$ algorithm form the TSP.